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2006 J. Phys. A: Math. Gen. 39 10849

(http://iopscience.iop.org/0305-4470/39/34/016)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **39** (2006) 10849–10857 doi:10.1088/0305-4470/39/34/016

Weyl correspondence and P-representation as operator Fredholm equations and their solutions

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Received 16 May 2006, in final form 14 July 2006
Published 9 August 2006
Online at stacks.iop.org/JPhysA/39/10849

Abstract
Using the technique of integration within an ordered product of operators we construct operator Fredholm equations, which can help us to re-formulize the Weyl correspondence and P-representation. We then search for their solutions which present new formulae for deriving quantum operators' Weyl classical correspondence and P-representation. As the application, we deduce some new relations about the two-variable Hermite polynomials.

PACS numbers: 03.65.Bz, 42.50.Dv

1. Introduction

Dirac's ket–bra operators are basic blocks in quantum representation theory [1]. By considering ket versus bra as special mathematical symbols that include non-commutative operators, in a recent paper (2006 *Ann. Phys.* **321** 480) [2] we have summarized the technique of integration within an ordered product (IWOP) of operators which enables Newton–Leibniz integration rules directly working for Dirac's ket–bra operators with continuum variables. For instance, using the coordinate eigenstate

$$|q\rangle = \pi^{-1/4} \exp\left(-\frac{q^2}{2} + \sqrt{2}qa^\dagger - \frac{a^{\dagger 2}}{2}\right)|0\rangle, \tag{1}$$

where $a^\dagger$ is the creation operator, $|0\rangle$ is the vacuum state annihilated by a, and $Q|q\rangle = q|q\rangle$, and the normally ordered form of vacuum projector $|0\rangle\langle 0| =: \mathrm{e}^{-a^\dagger a}:$, we can directly perform the integration

$$S_1 \equiv \int_{-\infty}^{\infty} \frac{\mathrm{d}q}{\sqrt{\mu}} \left|\frac{q}{\mu}\right\rangle \langle q| = \int_{-\infty}^{\infty} \frac{\mathrm{d}q}{\sqrt{\pi\mu}} \exp\left(-\frac{q^2}{2\mu^2} + \sqrt{2}\frac{q}{\mu}a^\dagger - \frac{a^{\dagger 2}}{2}\right)|0\rangle\langle 0|$$

$$\times \exp\left(-\frac{q^2}{2} + \sqrt{2}qa - \frac{a^2}{2}\right)$$

$$= \int_{-\infty}^{\infty} \frac{\mathrm{d}q}{\sqrt{\pi\mu}} : \exp\left(-\frac{q^2}{2}\left(1 + \frac{1}{\mu^2}\right) + \sqrt{2}q\left(\frac{a^\dagger}{\mu} + a\right) - \frac{1}{2}(a + a^\dagger)^2\right) :$$

$$= (\operatorname{sech}\lambda)^{1/2}\, \mathrm{e}^{-\frac{a^{\dagger 2}}{2}\tanh\lambda} : \mathrm{e}^{(\operatorname{sech}\lambda - 1)a^\dagger a} : \mathrm{e}^{\frac{a^2}{2}\tanh\lambda}, \qquad \mu = \mathrm{e}^\lambda. \tag{2}$$

Note that a commutes with $a^\dagger$ within : :, so $a^\dagger$ and a can be considered as if they were parameters while the integration is performed. Equation (2) is just the single-mode squeezing operator in normal ordering appearing in many references [3, 4]. It inspires a physical interpretation of some of the mathematical quantities employed in the theory: the classical dilation $q \to \frac{q}{\mu}$ maps into the normally ordered squeezing operator manifestly. It also exhibits that the fundamental representation theory can be formulated in not so abstract a way, as we can now directly perform the integral over ket–bra projection operators. Moreover, the IWOP technique can be employed to perform many complicated integrations for ket–bra projection operators.

In particular, for $\mu = 1$, equation (2) becomes the normally ordered Gaussian form

$$\int_{-\infty}^{\infty} \mathrm{d}q|q\rangle\langle q| = \int_{-\infty}^{\infty} \frac{\mathrm{d}q}{\sqrt{\pi}} : \exp\left(-q^2 + 2q\left(\frac{a + a^\dagger}{\sqrt{2}}\right) - \frac{1}{2}(a + a^\dagger)^2\right) :$$

$$= \int_{-\infty}^{\infty} \frac{\mathrm{d}q}{\sqrt{\pi}} : \mathrm{e}^{-(q-Q)^2} := 1, \qquad \text{(a real simple Gaussan integration!)} \tag{3}$$

where $Q = \frac{a+a^\dagger}{\sqrt{2}}$, $[a, a^\dagger] = 1$. In this work we shall show how the IWOP technique can help us to set up the normally ordered operator Fredholm equation [5] for Weyl–Wigner correspondence [6, 7] and P-representation [8, 9], respectively. We then search for these equations' solutions which present new formulae for deriving quantum operators' Weyl classical correspondence and P-representation. To illustrate what is a normally ordered operator Fredholm equation we take an example,

$$\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \mathrm{d}q : \mathrm{e}^{-(q-Q)^2} : \varphi(q) =: f(Q) :, \tag{4}$$

in which the normally ordered operator $: \exp[-(q - Q)^2] :$ is as an integral kernel. On the other hand, using (3) we have

$$\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \mathrm{d}q : \mathrm{e}^{-(q-Q)^2} : \varphi(q) = \int_{-\infty}^{\infty} \mathrm{d}q|q\rangle\langle q|\varphi(q) = \varphi(Q). \tag{5}$$

Comparing (4) and (5) we know the normally ordered expansion of $\varphi(Q)$ is

$$\varphi(Q) =: f(Q) :. \tag{6}$$

This is a new way to normally ordered expanding of an operator. We now search for the solution to the Fredholm equation (4); substituting the following expansions

$$: \mathrm{e}^{-(q-Q)^2} := \mathrm{e}^{-q^2} \sum_{n=0}^{\infty} : H_n(q)\frac{Q^n}{n!} :, \qquad \varphi(q) = \sum_{m=0}^{\infty} b_m H_m(q), \tag{7}$$

where

$$H_n(q) = 2^n \sum_{k=0}^{[n/2]} \frac{(-1)^k n!}{2^{2k} k!(n-2k)!} q^{n-2k}$$

is the single-variable Hermite polynomials, into (4) we have

$$\frac{1}{\sqrt{\pi}}\sum_{n,m=0}^{\infty}\int_{-\infty}^{\infty}\mathrm{d}q : \mathrm{e}^{-q^2}H_n(q)H_m(q)\frac{Q^n}{n!}b_m := \sum_{m=0}^{\infty}2^m b_m : Q^m :=: f(Q) :. \tag{8}$$

Taking the coherent state expectation value for (8), we see

$$\sum_{m=0}^{\infty}2^m b_m\langle z| : Q^m : |z\rangle = \sum_{m=0}^{\infty}2^m b_m(\sqrt{2}x)^m$$
$$= \langle z| : f(Q) : |z\rangle = f(\sqrt{2}x), \qquad z = x + \mathrm{i}y, \tag{9}$$

where $|z\rangle = \exp[-|z|^2/2 + za^\dagger]|0\rangle$ is the coherent state [8, 10]. After differentiating both sides of (9) m times with respect to $\sqrt{2}x$ and then setting $x = 0$, we obtain $f^{(m)}(0) = m!2^m b_m$; thus

$$\varphi(q) = \sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{2^n n!}H_n(q) \tag{10}$$

and

$$\varphi(Q) =: f(Q) := \sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{2^n n!}H_n(Q). \tag{11}$$

For instance, when in (4) $: f(Q) :=: Q^n :$, we see $f^{(m)}(0) = \delta_{n,m}m!$, so

$$: Q^n := \sum_{m=0}^{\infty}\frac{\delta_{n,m}m!}{2^m m!}H_m(Q) = \frac{1}{2^n}H_n(Q), \tag{12}$$

so $: 2^n Q^n :$ is the normally ordered expansion of $H_n(Q)$, which is an easily remembered operator formula. Equation (12) has many advantages in dealing with the properties of Hermite polynomials since $: Q^n :$ is more readily handled than $H_n(Q)$. For example, we can use (12) to easily check the generating function formula of Hermite polynomials

$$\sum_{n=0}^{\infty}\frac{t^n}{n!}H_n(Q) = \sum_{n=0}^{\infty}\frac{(2t)^n}{n!} : Q^n :=: \mathrm{e}^{2tQ} :=: \mathrm{e}^{\sqrt{2}t(a+a^\dagger)} := \mathrm{e}^{2tQ-t^2}. \tag{13}$$

Having experienced the simplest operator Fredholm equation (4) we, in section 2, shall set up operator Fredholm equations for Weyl correspondence and P-representation, respectively; as one can see later, in this way we can re-formulize these two theories and derive some new operator identities.

2. Operator Fredholm equation with use of $: \mathrm{e}^{-2(\alpha^*-a^\dagger)(\alpha-a)} :$

The Weyl correspondence rule [6] is a recipe for the quantization of functions defined in classical phase space. According to this rule, a classical function $h(q, p)$ corresponds to its quantum operator $H(Q, P)$ by the relation

$$H(Q, P) = \int\int_{-\infty}^{\infty}\mathrm{d}q\,\mathrm{d}p\Delta(q, p)h(q, p), \tag{14}$$

where the operators H, P and Q correspond to the classical quantities h, p and q, respectively, the Wigner operator $\Delta(p, q)$ is the integral kernel of the quantization scheme [7],

$$\Delta(p, q) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{d}u\,\mathrm{e}^{ipu}\left|q - \frac{u}{2}\right\rangle\left\langle q + \frac{u}{2}\right|, \tag{15}$$

where the state $\left|q-\frac{u}{2}\right\rangle$ is given by (1). Similarly to what we have done for (2), using the IWOP technique we perform the integral in (15) to yield its explicit Gaussian form [11]

$$\Delta(p,q)=\frac{1}{\pi}:\mathrm{e}^{-(p-P)^2-(q-Q)^2}:, \tag{16}$$

where the momentum operator P is related to Bose operators by $P=\frac{1}{\mathrm{i}\sqrt{2}}(a-a^\dagger)$. Equation (16) immediately leads to the correct marginal distributions $\int_{-\infty}^{\infty}\mathrm{d}p\Delta(q,p)=\frac{1}{\sqrt{\pi}}:\mathrm{e}^{-(q-Q)^2}:=|q\rangle\langle q|$ and $\int_{-\infty}^{\infty}\mathrm{d}q\Delta(q,p)=\frac{1}{\sqrt{\pi}}:\mathrm{e}^{-(p-P)^2}:=|p\rangle\langle p|$, where $|p\rangle$ is the momentum eigenvector. Due to

$$2\pi\,\mathrm{Tr}[\Delta(q,p)\Delta(q',p')]=\delta(q-q')\delta(p-p'), \tag{17}$$

it follows that

$$h(q,p)=2\pi\,\mathrm{Tr}[\Delta(q,p)H(Q,P)], \tag{18}$$

which is the usual formula to derive the classical Weyl correspondence of $H(Q,P)$.

Now we recast the Weyl correspondence theory into a new formalism, i.e., we construct an operator Fredholm equation for the Weyl correspondence and then search for its solution. In this way some properties of the two-variable Hermite polynomials can be directly derived.

Letting $\alpha=(q+\mathrm{i}p)/\sqrt{2}$, equation (16) becomes

$$\frac{1}{\pi}:\mathrm{e}^{-(p-P)^2-(q-Q)^2}:=\frac{1}{\pi}:\mathrm{e}^{-2(\alpha^*-a^\dagger)(\alpha-a)}:\equiv\Delta(\alpha,\alpha^*), \tag{19}$$

writing $h(q,p)\equiv g(\alpha,\alpha^*)$, then the Weyl correspondence (14) takes another form,

$$\begin{aligned}H(Q,P)\to G(a,a^\dagger)&=2\int\mathrm{d}^2\alpha\Delta(\alpha,\alpha^*)g(\alpha,\alpha^*)\\&=\frac{2}{\pi}\int\mathrm{d}^2\alpha:\mathrm{e}^{-2(\alpha^*-a^\dagger)(\alpha-a)}:g(\alpha,\alpha^*).\end{aligned} \tag{20}$$

When we perform the integration within : : in (20) with the result

$$G(a,a^\dagger)=:F(a,a^\dagger):, \tag{21}$$

then we set up

$$\frac{2}{\pi}\int\mathrm{d}^2\alpha:\mathrm{e}^{-2(\alpha^*-a^\dagger)(\alpha-a)}:g(\alpha,\alpha^*)=:F(a,a^\dagger):, \tag{22}$$

which is a normally ordered integration equation (the Fredholm equation of the first kind [5] with the kernel being $:\mathrm{e}^{-2(\alpha^*-a^\dagger)(\alpha-a)}:$). Instead of using (18) we aim to derive $g(\alpha,\alpha^*)$ from the given normally ordered operator $:F(a,a^\dagger):$ by solving equation (22). The advantage in doing so lies in that some new relations about the two-variable Hermite polynomials can naturally be deduced.

3. New formula for deriving Weyl's classical correspondence

In [12] we have shown that the generalized Bargmann representation of the two-mode number state $|m,n\rangle$ is

$$|m,n\rangle=\frac{a^{\dagger m}b^{\dagger n}}{\sqrt{m!n!}}|00\rangle\to\frac{1}{\sqrt{m!n!}}H_{m,n}(\xi,\xi^*)\,\mathrm{e}^{-\frac{1}{2}|\xi|^2}, \tag{23}$$

where $H_{m,n}(\xi, \xi^*)$ is the two-variable Hermite polynomial [13]

$$H_{m,n}(\xi, \xi^*) = \sum_{l=0}^{\min(m,n)} \frac{m!n!}{l!(m-l)!(n-l)!}(-1)^l \xi^{m-l} \xi^{*n-l},$$
$$[H_{m,n}(\xi, \xi^*)]^* = H_{m,n}(\xi^*, \xi) = H_{n,m}(\xi, \xi^*), \tag{24}$$

which is not a direct product of two independent single-variable Hermite polynomials. We say that $H_{m,n}(\xi, \xi^*)$ is the basis of the generalized Bargmann space because it spans an orthonormal and complete function space,

$$2\int\int \frac{d^2\xi}{\pi} e^{-2|\xi|^2} H_{m,n}(\sqrt{2}\xi, \sqrt{2}\xi^*)[H_{m',n'}(\sqrt{2}\xi, \sqrt{2}\xi^*)]^* = \sqrt{m!n!m'!n'!}\delta_{m,m'}\delta_{n,n'}, \tag{25}$$

so we can expand

$$g(\alpha, \alpha^*) = \sum_{m,n=0}^{\infty} C_{m,n} H^*_{m,n}(\sqrt{2}\alpha, \sqrt{2}\alpha^*), \tag{26}$$

where $C_{m,n}$ is the expansion coefficient to be determined. On the other hand, using the generating function of $H_{m,n}(\lambda, \lambda^*)$,

$$\sum_{m,n=0}^{\infty} \frac{t^m t'^n}{m!n!} H_{m,n}(\lambda, \lambda^*) = \exp\{-tt' + t\lambda + t'\lambda^*\}, \tag{27}$$

$$\text{or} \quad H_{m,n}(\lambda, \lambda^*) = \frac{\partial^m}{\partial t^m}\frac{\partial^n}{\partial t'^n} \exp\{-tt' + t\lambda + t'\lambda^*\}|_{t=t'=0}, \tag{28}$$

we can expand the normally ordered form of $\Delta(\alpha, \alpha^*)$ in (19) as

$$\Delta(\alpha, \alpha^*) = \frac{1}{\pi} e^{-2|\alpha|^2} : \sum_{m,n=0}^{\infty} \frac{(\sqrt{2}a^\dagger)^m(\sqrt{2}a)^n}{m!n!} H_{m,n}(\sqrt{2}\alpha, \sqrt{2}\alpha^*) :$$
$$= \frac{1}{\pi} e^{-2|\alpha|^2} : \sum_{m,n=0}^{\infty} \sqrt{2^{(m+n)}} \frac{a^{\dagger m} a^n}{m!n!} H_{m,n}(\sqrt{2}\alpha, \sqrt{2}\alpha^*) :. \tag{29}$$

Substituting (26) and (29) into the normally ordered Fredholm equation (22) and using (25) we have

$$(22) = \int \frac{2d^2\alpha}{\pi} e^{-2|\alpha|^2} : \sum_{m,n=0}^{\infty} \sqrt{2^{(m+n)}} \frac{a^{\dagger m} a^n}{m!n!} H_{m,n}(\sqrt{2}\alpha, \sqrt{2}\alpha^*)$$
$$\times : \sum_{m',n'=0}^{\infty} C_{m',n'} H^*_{m',n'}(\sqrt{2}\alpha, \sqrt{2}\alpha^*)$$
$$=: \sum_{m,n=0}^{\infty} C_{m,n}\sqrt{2^{(m+n)}} a^{\dagger m} a^n :=: F(a, a^\dagger) :. \tag{30}$$

Taking the coherent state expectation values of (30), we see

$$\langle z| : \sum_{m,n=0}^{\infty} C_{m,n}\sqrt{2^{(m+n)}} a^{\dagger m} a^n : |z\rangle = \langle z| : F(a, a^\dagger) : |z\rangle, \tag{31}$$

which is

$$\sum_{m,n=0}^{\infty} C_{m,n}\sqrt{2^{(m+n)}} z^{*m} z^n = F(z, z^*), \tag{32}$$

so

$$C_{m,n} = \frac{\partial^m \partial^n}{\sqrt{2^{(m+n)}} m! n! \partial z^{*m} \partial z^n} F(z, z^*)|_{z=0}. \tag{33}$$

Substituting (33) into (26) we obtain the solution of the Fredholm equation when $\langle z| : F(a, a^\dagger) : |z\rangle = F(z, z^*)$ is known,

$$g(\alpha, \alpha^*) = \sum_{m,n=0}^{\infty} \frac{1}{m! n! \sqrt{2^{(m+n)}}} H^*_{m,n}(\sqrt{2}\alpha, \sqrt{2}\alpha^*) \frac{\partial^m}{\partial z^{*m}} \frac{\partial^n}{\partial z^n} F(z, z^*)|_{z=0}. \tag{34}$$

This is a new formula for deriving Weyl's classical correspondence of normally ordered quantum operators. For example, when $: F_1(a, a^\dagger) := a^{\dagger m} a^n$, from (34) we have

$$g_1(\alpha, \alpha^*) = \frac{1}{\sqrt{2^{(m+n)}}} H^*_{m,n}(\sqrt{2}\alpha, \sqrt{2}\alpha^*). \tag{35}$$

So (22) takes the form

$$\frac{2}{\pi} \int \mathrm{d}^2\alpha : \mathrm{e}^{-2(\alpha^* - a^\dagger)(\alpha - a)} : H^*_{m,n}(\sqrt{2}\alpha, \sqrt{2}\alpha^*) = \sqrt{2^{(m+n)}} a^{\dagger m} a^n, \tag{36}$$

which enlightens us to obtain a new integration formula about $H_{m,n}$,

$$\int \frac{\mathrm{d}^2\xi}{\pi} \mathrm{e}^{-(\xi^* - \varsigma^*)(\xi - \varsigma)} H^*_{m,n}(\xi, \xi^*) = (\varsigma^*)^m \varsigma^n. \tag{37}$$

This is a non-trivial generalization of the mathematical formula [14, 18]

$$\int_{-\infty}^{\infty} \mathrm{d}x\, \mathrm{e}^{-(x-y)^2} H_n(x) = \sqrt{\pi} (2y)^n. \tag{38}$$

Thus we know the Weyl correspondence of $a^{\dagger m} a^n$ is $\frac{1}{\sqrt{2^{m+n}}} H^*_{m,n}(\sqrt{2}\alpha, \sqrt{2}\alpha^*)$,

$$\frac{1}{\sqrt{2^{m+n}}} H^*_{m,n}(\sqrt{2}\alpha, \sqrt{2}\alpha^*) = 2\pi\, \mathrm{Tr}[a^{\dagger m} a^n \Delta(\alpha, \alpha^*)]. \tag{39}$$

The correctness of (34) can be confirmed by substituting (34) into the Weyl correspondence formula (22), which exhibits

$$\sum_{m,n=0}^{\infty} \frac{1}{m! n! \sqrt{2^{m+n}}} \int \frac{2\mathrm{d}^2\alpha}{\pi} : \mathrm{e}^{-2(\alpha^* - a^\dagger)(\alpha - a)} : H^*_{m,n}(\sqrt{2}\alpha, \sqrt{2}\alpha^*) \times \frac{\partial^m}{\partial z^{*m}} \frac{\partial^n}{\partial z^n} F(z, z^*)|_{z=0} =: F(a, a^\dagger) :, \tag{40}$$

and then using (37) we see that the left-hand side of (40) becomes

$$\sum_{m,n=0}^{\infty} \frac{1}{m! n!} : a^{\dagger m} a^n : \frac{\partial^m}{\partial z^{*m}} \frac{\partial^n}{\partial z^n} F(z, z^*)|_{z=0} = : \exp\left\{ a^\dagger \frac{\partial}{\partial z^*} + a \frac{\partial}{\partial z} \right\} : F(z, z^*)|_{z=0} =: F(a, a^\dagger) :, \tag{41}$$

the right-hand side means that an operator's coherent state expectation value $F(z, z^*)$ can decide the operator itself, which is a known result [15, 16], since the coherent states are overcomplete. Hence the solution (35) is correct.

As a by-product, we see that Weyl correspondence (20)–(21) can be recast into the form (41), which seems new.

3.1. P-representation as an operator Fredholm equation—deriving $P(z)$ from ρ

Glauber [8] and Sudarshan [9] used the overcomplete set of coherent state $|z\rangle$ [10] to introduce the diagonal representation of the density matrix

$$\rho(a, a^\dagger) = \int \frac{\mathrm{d}^2 z}{\pi} P(z)|z\rangle\langle z|. \tag{42}$$

Though the $P(z)$ function (named as P-representation) has found widespread applications in quantum optics, it cannot be interpreted as a genuine probability distribution because it may take on negative values or become highly singular. The reverse relation of (42) is Mehta's formula [17]

$$P(z) = \frac{1}{\pi} \mathrm{e}^{|z|^2} \int \langle -\beta|\, \rho\, |\beta\rangle \exp[|\beta|^2 - \beta z^* + \beta^* z]\, \mathrm{d}^2\beta, \tag{43}$$

where $|\beta\rangle$ is also a coherent state. In this section, using $|z\rangle\langle z| =: \exp[-(z^* - a^\dagger)(z - a)] :$, we can rewrite (42) as

$$\rho(a, a^\dagger) = \int \frac{\mathrm{d}^2 z}{\pi} P(z) : \exp[-(z^* - a^\dagger)(z - a)] :, \tag{44}$$

i.e., the density matrix is a Gaussian convolution of the P function within : :. When we perform the integration within : : in (44) and find its result,

$$\rho(a, a^\dagger) =: F(a, a^\dagger) :, \tag{45}$$

then we can have

$$\frac{1}{\pi} \int \mathrm{d}^2\alpha : \exp[-(\alpha^* - a^\dagger)(\alpha - a)] : P(\alpha) =: F(a, a^\dagger) :, \tag{46}$$

which is a normally ordered Fredholm equation of the first kind with the kernel : $\mathrm{e}^{-(\alpha^* - a^\dagger)(\alpha - a)}$:. We aim to derive $P(\alpha)$ from the given operator : $F(a, a^\dagger)$: by solving equation (46).

For this purpose, we expand the P function as

$$P(\alpha) = \sum_{m,n=0}^{\infty} C'_{m,n} H^*_{m,n}(\alpha, \alpha^*), \tag{47}$$

where $C'_{m,n}$ is the expansion coefficient to be determined. On the other hand, using (27), we can expand : $\mathrm{e}^{-(\alpha^* - a^\dagger)(\alpha - a)}$: as

$$: \mathrm{e}^{-(\alpha^* - a^\dagger)(\alpha - a)} :=: \mathrm{e}^{-|\alpha|^2} \sum_{m,n=0}^{\infty} \frac{a^{\dagger m} a^n}{m!n!} H_{m,n}(\alpha, \alpha^*) :. \tag{48}$$

Substituting (47) and (48) into equation (46) and using (25) we have

$$(45) \to: \int \frac{\mathrm{d}^2\alpha}{\pi} \mathrm{e}^{-|\alpha|^2} \sum_{m,n=0}^{\infty} \frac{a^{\dagger m} a^n}{m!n!} H_{m,n}(\alpha, \alpha^*) \sum_{m',n'=0}^{\infty} C'_{m',n'} H^*_{m',n'}(\alpha, \alpha^*) :$$
$$= \sum_{m,n=0}^{\infty} C'_{m,n} : a^{\dagger m} a^n := \rho(a, a^\dagger). \tag{49}$$

Taking the coherent state expectation values of (49) we see

$$\langle z| : \sum_{m,n=0}^{\infty} C'_{m,n} a^{\dagger m} a^n : |z\rangle = \langle z| : F(a, a^\dagger) : |z\rangle \tag{50}$$

which is

$$\sum_{m,n=0}^{\infty} C'_{m,n} z^{*m} z^n = F(z, z^*), \tag{51}$$

so

$$C'_{m,n} = \frac{\partial^m \partial^n}{m!n!\partial z^{*m} \partial z^n} F(z, z^*)|_{z=0}. \tag{52}$$

Substituting (52) into (47) we obtain the solution of Fredholm equation when $\langle z| : F(a, a^\dagger) : |z\rangle = F(z, z^*)$ is known,

$$P(\alpha) = \sum_{m,n=0}^{\infty} \frac{1}{m!n!} H^*_{m,n}(\alpha, \alpha^*) \frac{\partial^m}{\partial z^{*m}} \frac{\partial^n}{\partial z^n} F(z, z^*)|_{z=0}, \tag{53}$$

or using (27), we have

$$P(\alpha) = \exp\left\{-\frac{\partial^2}{\partial z \partial z^*} + \alpha^* \frac{\partial}{\partial z^*} + \alpha \frac{\partial}{\partial z}\right\} F(z, z^*)|_{z=0}. \tag{54}$$

This is a new formula for deriving the P function when $F(z, z^*)$ is known, which differs from (43). For example, when $\rho = a^{\dagger m} a^n =: a^{\dagger m} a^n :$, from (53) we see

$$P(\alpha) = H^*_{m,n}(\alpha, \alpha^*) \tag{55}$$

which implies that the anti-normally ordered expansion of $a^{\dagger m} a^n$ is

$$\vdots H^*_{m,n}(a, a^\dagger) \vdots = a^{\dagger m} a^n, \tag{56}$$

where $\vdots\ \vdots$ denoted anti-normally ordering. For another example, when ρ is a pure coherent state $|\gamma\rangle\langle\gamma|$, using (53) and (27) and $\langle z|\gamma\rangle\langle\gamma|z\rangle = \mathrm{e}^{-(z^*-\gamma^*)(z-\gamma)}$ we have

$$\begin{aligned} P(\alpha) &= \sum_{m,n=0}^{\infty} \frac{1}{m!n!} H^*_{m,n}(\alpha, \alpha^*) \frac{\partial^m}{\partial z^{*m}} \frac{\partial^n}{\partial z^n} \mathrm{e}^{-(z^*-\gamma^*)(z-\gamma)}|_{z=0} \\ &= \mathrm{e}^{-\gamma^*\gamma} \sum_{m,n=0}^{\infty} \frac{1}{m!n!} H^*_{m,n}(\alpha, \alpha^*) H_{m,n}(\gamma, \gamma^*). \end{aligned} \tag{57}$$

Using another completeness relation regarding the two-variable Hermite polynomials

$$\sum_{m,n=0}^{\infty} \frac{1}{m!n!} H^*_{m,n}(\alpha, \alpha^*) H_{m,n}(\gamma, \gamma^*) = \pi\delta(\gamma^* - \alpha^*)\delta(\gamma - \alpha)\, \mathrm{e}^{\gamma^*\gamma}, \tag{58}$$

we know that the P-representation of $|\gamma\rangle\langle\gamma|$ is $P(\alpha) = \pi\delta(\gamma^* - \alpha^*)\delta(\gamma - \alpha)$, as expected. Note that equation (58) can also be derived by using the integration form of $H_{m,n}(\alpha, \alpha^*)$ [16].

To sum up, using the IWOP technique we have constructed the operator Fredholm equations for Weyl correspondence and P-representation; we then derive their solutions which provide a new formula for deriving Weyl's classical correspondence and P-representation. In this way, some properties of the two-variable Hermite polynomials can easily be derived. We wish this paper can enrich the fundamental representation theory of the quantum light field, as readers might see that using two-variable (multi-variable) Hermite polynomials the bipartite (multi-particle) entangled state representation can be constructed; thus one has the possibility of having a statistical description of an electromagnetic field in terms of various entangled state representations.

Acknowledgments

The authors sincerely thank the referees for their careful reading and suggestions. This work was supported by the National Natural Science Foundation of China under grant 10475056 and Specialized Research Fund for the Doctorial Progress (SRFDP) of Higher Education in China.

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